

The Larson-Sweedler theorem for multiplier Hopf algebras

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Abstract

Any finite-dimensional Hopf algebra has a left and a right integral. Conversely, Larsen and Sweedler showed that, if a finite-dimensional algebra with identity and a comultiplication with counit has a faithful left integral, it has to be a Hopf algebra.

In this paper, we generalize this result to possibly infinite-dimensional algebras, with or without identity. We have to leave the setting of Hopf algebras and work with multiplier Hopf algebras. Moreover, whereas in the finite-dimensional case, there is a complete symmetry between the bialgebra and its dual, this is no longer the case in infinite dimensions. Therefore we consider a direct version (with integrals) and a dual version (with cointegrals) of the Larson-Sweedler theorem.

We also add some results about the antipode. Furthermore, in the process of this paper, we obtain a new approach to multiplier Hopf algebras with integrals.

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0. Introduction

Let A be an algebra over \mathbb{C} with identity 1. Let $\Delta : A \rightarrow A \otimes A$ be a comultiplication on A . So Δ is a unital homomorphism and satisfies coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (where we use ι to denote the identity map). Let us also assume the existence of a counit. It is a homomorphism $\varepsilon : A \rightarrow \mathbb{C}$ such that $(\varepsilon \otimes \iota)\Delta(a) = a$ and $(\iota \otimes \varepsilon)\Delta(a) = a$ for all $a \in A$.

Such a pair (A, Δ) is a Hopf algebra if there also exists an antipode. This is a anti-homomorphism $S : A \rightarrow A$ such that $m(S \otimes \iota)\Delta(a) = \varepsilon(a)1$ and $m(\iota \otimes S)\Delta(a) = \varepsilon(a)1$ for all $a \in A$ (where we use m to denote the multiplication map, defined as a linear map from $A \otimes A$ to A by $m(a \otimes b) = ab$).

It is well known that any finite-dimensional Hopf algebra has a left and a right integral (see [A], [S] and [VD4] for an alternative approach). Recall that a left integral is a non-zero linear map $\varphi : A \rightarrow \mathbb{C}$ such that $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$ while a right integral is a non-zero linear map $\psi : A \rightarrow \mathbb{C}$ such that $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all $a \in A$. These integrals are faithful in the sense that the bilinear forms $(a, b) \mapsto \varphi(ab)$ and $(a, b) \mapsto \psi(ab)$ are non-degenerate (see e.g. Proposition 3.4 in [VD5] for a general proof of this fact).

Conversely, Larson and Sweedler showed in [L-S] that a pair (A, Δ) of a finite-dimensional algebra with 1 and a comultiplication with a counit ε is actually a Hopf algebra if there exists a faithful left integral. Their result is stated in its *dual form*, as we will explain in Section 4 of this paper.

In this paper we treat two generalizations of this result. In the first place, we no longer restrict to the finite-dimensional case. The price we have to pay is that we need to assume the existence of both a left and a right integral. Also because we no longer assume the underlying algebra to be finite-dimensional, there is a difference between the direct and the dual version. The first one is about integrals (Section 2) while the second one assumes the existence of cointegrals (Section 4). It also seems to be more natural (and also more general) to leave the framework of Hopf algebras (with unital algebras) and pass to the more general theory of multiplier Hopf algebras (where the algebras may not have an identity). We will also briefly mention the $*$ -algebra case but this is of minor importance for the treatment here.

The results, as well as the methods used to prove these results, are greatly inspired by the recent developments in the theory of locally compact quantum groups (see e.g. [K-V1] and [K-V2]). This theory can be considered as the operator algebra approach to quantum groups. Before this theory was developed, there was an obvious need for an operator algebra version of the notion of a Hopf algebra. In operator algebras, people are familiar with algebras without identity and the notion of a multiplier algebra is commonly used. This inspired the first author to generalize the notion of a Hopf algebra to algebras possibly without identity and this lead to the theory of multiplier Hopf algebras (see [VD2]).

But there is more. While the antipode in a purely algebraic setting is not so hard to handle, it always has been a troublesome object in the operator algebra approach. To begin with, in interesting cases, the antipode becomes an unbounded map, not everywhere defined. Moreover, basic formulas like $m(S \otimes \iota)\Delta(a) = \varepsilon(a)1$ become very hard to generalize (for

various reasons). This explains why, in the operator algebra approach to quantum groups, usually the antipode is not part of the axioms but rather an object that is constructed.

The most striking example of this we find in [K-V1, K-V2] where a theory of locally compact quantum groups is developed. Roughly speaking, a locally compact quantum group is defined as an operator algebra with a comultiplication satisfying certain conditions. One of these conditions is the existence of integrals. Then from these axioms, the antipode is constructed.

An other example is found in [V-VD] where a theory of Hopf C^* -algebras is developed. In this case, however, no integrals are involved. Roughly speaking, this is the operator algebra version of the theory of multiplier Hopf algebras, as developed in [VD2]. Also in this theory, the antipode is not part of the axioms but it is constructed.

The case of a locally compact quantum group is more like the situation of the Larson-Sweedler theorem for finite-dimensional Hopf algebras, but also the two other cases, where also the antipode is constructed, have been sources of inspiration for this paper. So, we have used some of the ideas and techniques from these works on the operator algebra approach to quantum groups in this paper. This again is a nice illustration of the advantage of having different approaches to quantum groups and the importance of the mutual influence.

Let us mention here the fact that the Larson-Sweedler result was first obtained by Kac and Palyutkin in the special case of a finite-dimensional operator algebra ([K-P]). See a remark in the introduction of [L-S].

We work in a purely algebraic setting however and no knowledge about the operator algebra approach to quantum groups is needed. So we will not use results from e.g [V-VD], nor from [K-VD1, K-VD2]. We will just adopt some of the techniques to this purely algebraic context. So, we mainly aim at algebraists but it may be interesting for the reader to know that we have some influence coming from analysis.

Let us now give a short survey of *the content of this paper* and be a bit more precise about the main results and their relation with the original work by Larson and Sweedler.

At the end of this introduction, we will give some basic references and fix some notations used throughout this paper. *Section 1* is devoted to some preliminaries. We recall the definition of a comultiplication Δ on an algebra A possibly without identity. We also recall the notion of a (regular) multiplier Hopf algebra. New is the definition of the legs of Δ and the concept of a regular and of a full comultiplication.

In *Section 2* we prove the main result of the paper. We take an algebra with a comultiplication. We assume the existence of integrals and we show that we must have a multiplier Hopf algebra. We also give another proof of (the dual form of) the original theorem of Larson and Sweedler for a finite-dimensional algebra with a unit, a comultiplication and a counit.

In *Section 3* we discuss properties of the counit and the antipode. Recall that in the definition of a multiplier Hopf algebra, the counit and the antipode are not present, they are constructed. In this section we will give a new way to construct these objects, using the integrals. As a byproduct, we obtain a different approach to multiplier Hopf algebras with integrals (sometimes called algebraic quantum groups, cf. [VD5]).

In *Section 4*, we treat the case of a cointegral. We again take an algebra with a comultiplication. We first define the notion of a left cointegral. We do not need a counit to define this concept. In fact, our definition of a left cointegral makes it possible to construct the counit. Then we assume the existence of a left and of a right cointegral and again we show that we have a multiplier Hopf algebra (of discrete type, cf. [VD5, VD-Z1]). Also here we discuss the relation of the antipode with the cointegrals. Specializing, to the finite-dimensional case, we again find a proof of (the original form of) the Larson-Sweedler theorem as it is found in [L-S].

We finish this introduction by collecting some of the *basic notions* for this paper and give some standard references.

We work with algebras over \mathbb{C} with or without identity. If the algebra has an identity, we denote it by 1. If there is no identity, we want the product to be non-degenerate, i.e. if $x \in A$ then $x = 0$ if either $ax = 0$ for all $a \in A$ or $xa = 0$ for all $a \in A$. The product in an algebra with identity is automatically non-degenerate. For any algebra A we have the multiplier algebra $M(A)$. It can be characterized as the largest algebra with identity such that A sits in $M(A)$ as an essential two-sided ideal. By *essential* we mean that if $x \in M(A)$ then $x = 0$ if either $xa = 0$ for all $a \in A$ or $ax = 0$ for all $a \in A$. If A has an identity, then $M(A) = A$.

If A and B are algebras as above and $\gamma : A \rightarrow M(B)$ is an algebra map, then γ is called non-degenerate if $\gamma(A)B = B\gamma(A) = B$. If this is the case, then γ has a unique extension to a unital homomorphism from $M(A)$ to $M(B)$. This extension is still denoted by γ . For details, see e.g. the appendix in [VD2].

We will use the tensor product $A \otimes A$ for an algebra A . It is again an algebra with a non-degenerate product. So we can also define the multiplier algebra $M(A \otimes A)$. A comultiplication on A is a homomorphism from A to $M(A \otimes A)$. All the comultiplications we encounter in this paper will turn out to be non-degenerate. So they can be extended to unital maps from $M(A)$ to $M(A \otimes A)$. Then coassociativity can be stated in the form $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ where ι is the identity map and $\Delta \otimes \iota$ and $\iota \otimes \Delta$ are the extensions of the corresponding homomorphisms from $A \otimes A$ to $M(A \otimes A \otimes A)$. There is also another way to look at coassociativity (see Definition 1.1 in Section 1 of this paper).

Occasionally, we will use the Sweedler notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and $(\Delta \otimes \iota)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. The use of the Sweedler notation in the case of multiplier Hopf algebras has been justified e.g. in [D-VD] but it has to be done with some care.

Let us now collect some *standard references*. For the general theory of Hopf algebras, we refer to [A] and [S]. The original reference for multiplier Hopf algebras is [VD2] but a survey can be found in [VD-Z2]. Integrals on multiplier Hopf algebras have been studied in [VD5] and they are also considered in the survey paper [VD-Z2]. Multiplier Hopf algebras with cointegrals (i.e. multiplier Hopf algebras of discrete type) were already introduced in [VD5] but have been studied in [VD-Z1].

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1. Preliminaries

Let A be an algebra over \mathbb{C} , with or without unit, but with a non-degenerate product. Consider the tensor product $A \otimes A$ of A with itself. The product in $A \otimes A$ will again be non-degenerate. We can consider the multiplier algebras $M(A)$ and $M(A \otimes A)$ of A and $A \otimes A$ respectively. There are natural imbeddings of $A \otimes A$ in $M(A) \otimes M(A)$ and of $M(A) \otimes M(A)$ in $M(A \otimes A)$. In general, these two imbeddings are strict.

This is the setting we need to define what we will mean by a (regular) comultiplication in this paper (see e.g. [VD2]):

1.1 Definition A homomorphism $\Delta : A \rightarrow M(A \otimes A)$ is called a *comultiplication* if

- i) $\Delta(a)(1 \otimes b) \in A \otimes A$ and $(a \otimes 1)\Delta(b) \in A \otimes A$ for all $a, b \in A$,
- ii) $(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$ for all $a, b, c \in A$ (*coassociativity*).

Observe that i) is needed to give a meaning to ii). As we mentioned already in the introduction, we use 1 for the identity (here in $M(A)$) and ι for the identity map (here from A to itself). If A has an identity, then i) is automatic and ii) is nothing else but coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

As we have mentioned in the introduction, occasionally, we will say something about the $*$ -algebra case. When we have a comultiplication Δ on a $*$ -algebra, we always require Δ to be a $*$ -homomorphism.

In this paper, we will only work with regular comultiplications:

1.2 Definition A comultiplication Δ on A is called *regular* if also

- iii) $\Delta(a)(b \otimes 1) \in A \otimes A$ and $(1 \otimes a)\Delta(b) \in A \otimes A$ for all $a, b \in A$.

For an algebra with identity, regularity is automatic. Also in the case of a $*$ -algebra, a comultiplication is automatically regular in the above sense.

We have the following result.

1.3 Proposition If Δ is a regular comultiplication, then the opposite comultiplication Δ' , obtained by composing Δ with the flip map, is again a comultiplication.

Proof: First recall the definition of the opposite comultiplication. The flip map $\sigma : A \otimes A \rightarrow A \otimes A$ is defined as usual by $\sigma(a \otimes b) = b \otimes a$. It can be extended uniquely

to a homomorphism, still denoted by σ , from $M(A \otimes A)$ to itself. Then Δ' is defined by $\Delta'(a) = \sigma(\Delta(a))$ for all $a \in A$.

Now, it is obvious that condition iii) in Definition 1.2 gives condition i) in Definition 1.1 for Δ' . To prove that Δ' is also coassociative, one simply has to start with the formula in ii), multiply with $1 \otimes 1 \otimes c'$ on the left and with $a' \otimes 1 \otimes 1$ on the right, bring where possible $1 \otimes c'$ and $a' \otimes 1$ inside the brackets, take $a \otimes 1$ and $1 \otimes c$ outside and finally use the non-degeneracy of the product and cancel $a \otimes 1 \otimes 1$ on the left and $1 \otimes 1 \otimes c$ on the right. This will yield condition ii) for Δ' provided we flip the first and the third factor.

Let us now recall the notion of a (regular) multiplier Hopf algebra (see [VD2]).

1.4 Definition Let A be an algebra over \mathbb{C} with a non-degenerate product and let Δ be a comultiplication on A . Then (A, Δ) is called a *multiplier Hopf algebra* if the linear maps T_1, T_2 , defined from $A \otimes A$ to $A \otimes A$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are bijective. A multiplier Hopf algebra is called *regular* if Δ is a regular comultiplication and if also the maps above, now for Δ' , are bijections.

Remark that any Hopf algebra is a multiplier Hopf algebra and conversely, any multiplier Hopf algebra, with an algebra with identity, is a Hopf algebra (see [VD2]). Regularity is not automatic for Hopf algebras, but equivalent with the antipode being bijective. This is automatic in the case of a multiplier Hopf $*$ -algebra (i.e. when the underlying algebra is a $*$ -algebra).

We will need the notion of *the legs* of Δ :

1.5 Definition Let A be an algebra with a non-degenerate product and Δ a comultiplication on A . For any $a \in A$ we define *the left leg* of $\Delta(a)$ as the smallest subspace V of A so that

$$\Delta(a)(1 \otimes A) \subseteq V \otimes A.$$

Similarly, *the right leg* of $\Delta(a)$ is the smallest subspace W of A so that

$$(A \otimes 1)\Delta(a) \subseteq A \otimes W.$$

Similarly, we define the left and the right legs of Δ as the smallest subspaces V and W of A satisfying

$$\Delta(A)(1 \otimes A) \subseteq V \otimes A \quad \text{and} \quad (A \otimes 1)\Delta(A) \subseteq A \otimes W.$$

In the case of a multiplier Hopf algebra, we clearly have that the left and the right leg of Δ are all of A (because the maps T_1 and T_2 are surjective). In fact using the counit ε , we see that a belongs to both the right and the left leg of $\Delta(a)$.

In general, we can show the following.

1.6 Proposition The left leg of $\Delta(a)$ is spanned by elements of the form $(\iota \otimes \omega)(\Delta(a)(1 \otimes b))$ where $b \in A$ and $\omega \in A'$ (the dual space of A). Similarly, the right leg of $\Delta(a)$ is spanned by the elements $(\omega \otimes \iota)((b \otimes 1)\Delta(a))$ where $b \in A$ and $\omega \in A'$.

Proof: It is clear that $(\iota \otimes \omega)(\Delta(a)(1 \otimes b))$ belongs to the left leg of $\Delta(a)$ for all $b \in A$ and $\omega \in A'$. Conversely, let V be the space spanned by these elements. We need to show that $\Delta(a)(1 \otimes b) \in V \otimes A$ for all $b \in A$. Take $b \in A$ and write $\Delta(a)(1 \otimes b) = \sum p_i \otimes q_i$ with the $\{q_i\}$ linearly independent. Choose $\omega \in A'$ such that $\omega(q_i) = 1$ for some i and $\omega(q_j) = 0$ for $j \neq i$. Then $p_i = (\iota \otimes \omega)(\Delta(a)(1 \otimes b))$. By assumption $p_i \in V$. This is true for all i and so $\Delta(a)(1 \otimes b) \subseteq V \otimes A$.

Similarly for the right leg of $\Delta(a)$.

For a regular comultiplication, we have the following.

1.7 Proposition If Δ is a regular comultiplication and $a \in A$, then the left leg of $\Delta(a)$ is also the smallest subspace V' of A so that

$$(1 \otimes A)\Delta(a) \subseteq V' \otimes A.$$

Proof: Suppose that V' is a subspace of A so that $(1 \otimes A)\Delta(a) \subseteq V' \otimes A$. We will show that $\Delta(a)(1 \otimes b) \in V' \otimes A$ for all $b \in A$. Then it will follow that the left leg of $\Delta(a)$ is contained in V' . A similar argument will give that V' is contained in the left leg of $\Delta(a)$ and this will prove the result.

So, let $b \in A$ and write $\Delta(a)(1 \otimes b) = \sum p_i \otimes q_i$ with the $\{q_i\}$ independent. We know that $\sum p_i \otimes cq_i = (1 \otimes c)\Delta(a)(1 \otimes b) \in V' \otimes A$ for all $c \in A$. Assume that $\omega \in A'$ and that $\omega(x) = 0$ for all $x \in V'$. Then $\sum \omega(p_i)cq_i = 0$ for all c . By the non-degeneracy of the product, we get $\sum \omega(p_i)q_i = 0$. As the $\{q_i\}$ are chosen to be linearly independent, it follows that $\omega(p_i) = 0$ for all i . This is true for any ω that vanishes on V' . Therefore $p_i \in V'$ for all i . Hence $\Delta(a)(1 \otimes b) \in V' \otimes A$. This proves the claim.

Of course, we have a similar result for the right leg of $\Delta(a)$ and similar results as in Proposition 1.6, now with elements $(\iota \otimes \omega)((1 \otimes b)\Delta(a))$ for the left leg and $(\omega \otimes \iota)(\Delta(a)(b \otimes 1))$ for the right leg. We also have similar results for the left and the right leg of Δ .

If A is a $*$ -algebra, then the comultiplication is automatically regular so that Proposition 1.7 applies. Then, if a is a self-adjoint element, i.e. $a = a^*$, the left and right legs of $\Delta(a)$ are self-adjoint subspaces of A .

We finish this preliminary section with the following definition.

1.8 Definition Let Δ be a comultiplication on A . We call it *full* if the left and the right legs of Δ are all of A .

As we have seen already, when (A, Δ) is a multiplier Hopf algebra, then Δ is full. In fact this will already be the case when there is a counit (i.e. a linear map ε so that $(\iota \otimes \varepsilon)\Delta(a) = a$,

in the sense that $(\iota \otimes \varepsilon)((b \otimes 1)\Delta(a)) = ba$ for all $a, b \in A$ and so that $(\varepsilon \otimes \iota)\Delta(a) = a$ in the sense that $(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab$.

We also have a full comultiplication if the maps T_1 and T_2 , defined as before by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are surjective. However, it should be noticed that, in general, the surjectivity of these maps is not implied when Δ is full. In the next section, we will see that this is the case when there exist integrals.

2. The Larson-Sweedler theorem

Throughout this section, A will be an algebra over \mathbb{C} , with a non-degenerate product and Δ will be a comultiplication on A . Most of the time, we will assume that Δ is regular.

Let us recall the definition of an integral in this setting.

2.1 Definition A linear functional φ on A is called *left invariant* if

$$(\iota \otimes \varphi)((b \otimes 1)\Delta(a)) = \varphi(a)b$$

for all $a, b \in A$. A non-zero left invariant functional is called a *left integral*. Similarly, a linear functional ψ is called *right invariant* if

$$(\psi \otimes \iota)(\Delta(a)(1 \otimes b)) = \psi(a)b$$

for all $a, b \in A$. A non-zero right invariant functional is a *right integral*.

In the case of a $*$ -algebra, it is common to assume positivity of these integrals. This means that $\varphi(a^*a) \geq 0$ for all a and similarly for ψ . In any case, one can always assume that $\varphi(a^*) = \overline{\varphi(a)}$ without loss of generality.

In the case of a regular comultiplication, for every $a \in A$ and $\omega \in A'$ (the dual space of A), we can define elements $(\iota \otimes \omega)\Delta(a)$ and $(\omega \otimes \iota)\Delta(a)$ in $M(A)$ in the obvious way. Then $\varphi \in A'$ is left invariant if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$ and $\psi \in A'$ is right invariant if $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all $a \in A$.

We will also use the following terminology.

2.2 Definition A linear functional ω on A is called *faithful* if the bilinear map $(a, b) \mapsto \omega(ab)$ is non-degenerate.

So ω is faithful if $b = 0$ when $\omega(ab) = 0$ for all a or when $\omega(ba) = 0$ for all a .

In the case of a finite-dimensional algebra, we only need one of these conditions. Indeed, suppose that $b \mapsto \omega(\cdot b)$ is injective from A to A' . Then it is also surjective. If now $\omega(ab) = 0$ for all $b \in A$, then $\rho(a) = 0$ for all $\rho \in A'$ and so $a = 0$.

In the case of a *-algebra and a positive linear functional ω , faithfulness means that $\omega(a^*a) = 0$ implies $a = 0$.

It was shown in [VD5, Proposition 3.4] that integrals on multiplier Hopf algebras are automatically faithful. They are also unique [VD5, Theorem 3.7].

Now we are able to formulate the main result.

2.3 Theorem Let A be an algebra with a non-degenerate product and assume that Δ is a full and regular comultiplication on A . If there exists a faithful left integral and a faithful right integral, then (A, Δ) is a regular multiplier Hopf algebra.

Later in this section, we will look at the finite-dimensional case and discuss this result. In particular, we will relate it with the original theorem of Larson and Sweedler. The reader who is interested in the deeper relation with the operator algebra approach to quantum groups, is invited to compare the assumptions in this theorem with the axioms of a locally compact quantum group in [K-V1] and [K-V2].

Now, we start with the proof of the theorem. We split up the proof in a few lemmas as we will need these smaller results later when we discuss the theorem in the finite-dimensional case.

2.4 Lemma Let Δ be any comultiplication on A . Assume that there is a faithful right integral. Then the linear map T_1 , defined by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$, is injective.

Proof: Suppose $\sum \Delta(a_i)(1 \otimes b_i) = 0$. Take any $x \in A$ and multiply with $\Delta(x)$ from the left and apply $\psi \otimes \iota$ where ψ is a right integral. We get, using the invariance, that $\sum \psi(xa_i)b_i = 0$. If now ψ is faithful, we get $\sum a_i \otimes b_i = 0$. This proves the injectivity of T_1 .

In a similar way, when Δ is regular, we get the injectivity of the map $a \otimes b \mapsto (1 \otimes a)\Delta(b)$ when we have a faithful right integral. When we have a faithful left integral, we get T_2 injective with $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$. And when Δ is regular, also $a \otimes b \mapsto \Delta(a)(b \otimes 1)$ will be injective.

In other words we get:

2.5 Proposition If Δ is a regular comultiplication and if there exist faithful left and right integrals, then the four maps

$$\begin{array}{ll} a \otimes b \mapsto \Delta(a)(1 \otimes b) & a \otimes b \mapsto \Delta(a)(b \otimes 1) \\ a \otimes b \mapsto (a \otimes 1)\Delta(b) & a \otimes b \mapsto (1 \otimes a)\Delta(b) \end{array}$$

are all injective.

In the *-algebra case, the injectivity of the maps on the left will already imply the injectivity of the maps on the right.

What about surjectivity? We get the following.

2.6 Lemma Let Δ be a regular comultiplication and let φ be a left integral. Take $a, b \in A$ and define $x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$. Then $x \otimes c$ belongs to the range of T_1 for all $c \in A$.

Proof: We claim that $x \otimes c = T_1(y)$ where

$$y = (\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1))$$

with $\Delta_{23}(b) = 1 \otimes \Delta(b)$ and $\Delta_{13}(a) = (1 \otimes \sigma)(\Delta(a) \otimes 1)$ where σ is the flip map.

First let us verify that y is well-defined in $A \otimes A$. Because Δ is assumed to be regular, we have $\Delta(b)(c \otimes 1) \in A \otimes A$. For all $p, q \in A$ we get $\Delta_{13}(a)(1 \otimes p \otimes q) \in A \otimes A \otimes A$. So $\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1) \in A \otimes A \otimes A$ and if we apply $\iota \otimes \iota \otimes \varphi$, we get $y \in A \otimes A$.

Now

$$(T_1 \otimes \iota)(\Delta_{13}(a)(1 \otimes p \otimes q)) = ((\Delta \otimes \iota)\Delta(a))(1 \otimes p \otimes q)$$

and so

$$\begin{aligned} (T_1 \otimes \iota)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1)) &= ((\Delta \otimes \iota)\Delta(a))((1 \otimes \Delta(b))(1 \otimes c \otimes 1)) \\ &= (\iota \otimes \Delta)(\Delta(a)(1 \otimes b))(1 \otimes c \otimes 1). \end{aligned}$$

If we apply $\iota \otimes \iota \otimes \varphi$, it will follow from the left invariance of φ that

$$T_1(y) = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) \otimes c = x \otimes c$$

and this proves the result.

Now we use this lemma to prove that T_1 is surjective under certain conditions.

2.7 Lemma Let Δ be a regular comultiplication such that the left leg of Δ is all of A . If there is a faithful left integral, then T_1 is surjective.

Proof: Because of the previous lemma, we have to show that A is spanned by the elements $(\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$ where $a, b \in A$. Suppose that $\omega \in A'$ and that $\omega((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = 0$ for all $a, b \in A$. Then $\varphi((\omega \otimes 1)(\Delta(a)(1 \otimes b))b') = 0$ for all $a, b, b' \in A$. Because φ is faithful, $(\omega \otimes 1)(\Delta(a)(1 \otimes b)) = 0$ for all $a, b \in A$. Then ω is 0 on the left leg of Δ . By assumption $\omega = 0$. It follows that indeed A is spanned by elements of the form $(\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$ with $a, b \in A$. This proves the lemma.

In a completely similar fashion, we will get that the following is true.

2.8 Proposition Let Δ be a regular and full comultiplication. Assume that there exist a faithful left integral and a faithful right integral. Then the four maps

$$\begin{array}{ll} a \otimes b \mapsto \Delta(a)(1 \otimes b) & a \otimes b \mapsto \Delta(a)(b \otimes 1) \\ a \otimes b \mapsto (a \otimes 1)\Delta(b) & a \otimes b \mapsto (1 \otimes a)\Delta(b) \end{array}$$

are surjective.

Again, in the $*$ -algebra case, the surjectivity of the maps on the left will imply already the surjectivity of the maps on the right.

Now, we have also completed the proof of the theorem. Indeed, if Δ is regular and full and if we have a faithful left integral and a faithful right integral, all these four maps are bijective. Then, (A, Δ) is a regular multiplier Hopf algebra (see [VD2]).

In the next section, we will see how the antipode and the counit can be obtained in this case, without using the general theory.

Now, we finish this section by looking at the finite-dimensional case.

2.9 Theorem Let A be a finite-dimensional algebra with 1 and Δ a comultiplication on A . Assume that the left leg of Δ is all of A . If there is a faithful left integral, then (A, Δ) is a Hopf algebra.

Proof: By the analogue of Lemma 2.4, the map T_2 , defined by $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$ is injective. Because we are working in finite dimensions, this map is also surjective. By Lemma 2.7 the map T_1 , defined by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ is surjective. Again here this implies that it is also injective. Hence (A, Δ) is a Hopf algebra.

If we have a finite-dimensional algebra A with 1 and a comultiplication Δ with a counit ε , then this comultiplication is automatically full. So in this case the extra assumption in the theorem above (about the left leg of Δ) is fulfilled. Then we get a (dual) version of the original Larson-Sweedler theorem, as we find it in [L-S]. The faithfulness of the left integral in our result corresponds with the non-singularity assumption in their theorem. In Section 4 we will recover the Larson-Sweedler as it is formulated in the original paper (and we will give some more comments).

3. The counit and the antipode

In [VD2], we have constructed the counit and the antipode for a multiplier Hopf algebra. Let us recall how this was done (see Section 3 and 4 in [VD2]).

3.1 Proposition Let (A, Δ) be a multiplier Hopf algebra. There exists a linear map $\varepsilon : A \rightarrow \mathbb{C}$ defined by $\varepsilon(a)b = \sum p_i q_i$ when $a \otimes b = \sum \Delta(p_i)(1 \otimes q_i)$. There exists a linear map $S : A \rightarrow M(A)$ defined by $S(a)b = \sum \varepsilon(p_i)q_i$ when $a \otimes b = \sum \Delta(p_i)(1 \otimes q_i)$.

The bijectivity of the map T_1 , as defined already in Section 2 by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$, is one of the assumptions of a multiplier Hopf algebra and is used in the proof of the

proposition above. If the multiplier Hopf algebra is regular, it is shown that S maps A into A and that it is bijective.

In the case where we have a regular multiplier Hopf algebra with a left integral, we have the following result.

3.2 Proposition If (A, Δ) is a regular multiplier Hopf algebra and φ a left integral, then

$$\varepsilon((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = \varphi(ab)$$

and

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b))$$

for all $a, b \in A$.

The first formula follows trivially from the property of the counit saying

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab.$$

Also the second formula can be proven from the standard properties of the antipode.

However, here let us use the formula obtained in Lemma 2.6. We showed that

$$x \otimes c = T_1((\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1)))$$

where $x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$. Then it follows already from the defining formulas in Proposition 3.1 that

$$\varepsilon(x)c = (\iota \otimes \varphi)(\Delta(a)\Delta(b)(c \otimes 1)) = \varphi(ab)c,$$

so that $\varepsilon(x) = \varphi(ab)$, and

$$\begin{aligned} S(x)c &= (\varepsilon \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1)) \\ &= (\iota \otimes \varphi)((1 \otimes a)\Delta(b)(c \otimes 1)) \end{aligned}$$

so that $S(x) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b))$.

In the remaining of this section, as we announced already in the introduction, we will use the formulas in Proposition 3.2 to construct the counit and the antipode when we have a pair of an algebra A with a coproduct as in Theorem 2.3 and to prove their properties. We do not get any new results but it seems nice and instructive. We do not use any of the results from [VD5]. We will mainly take advantage of the extra information we get from the integrals. So, we get an alternative approach to the theory of multiplier Hopf algebras with integrals (the so-called algebraic quantum groups). Whenever convenient, we feel free to use the classical arguments.

So, as in Theorem 2.3, let A be an algebra over \mathbb{C} , with a non-degenerate product, and let Δ be a regular comultiplication which is full, that is such that the two legs of Δ are

all of A . Also assume that we have a faithful left integral φ and a faithful right integral ψ . We know that these integrals must be unique (up to a scalar), but we will not use this information.

First, we treat the counit. To define the counit, we need the following lemma.

3.3 Lemma If $\sum(\iota \otimes \varphi)(\Delta(a_i)(1 \otimes b_i)) = 0$, then also $\sum \varphi(a_i b_i) = 0$.

Proof: We know that for all $a, b, c \in A$ we have

$$x \otimes c = T_1((\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1)))$$

where $x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$. From the existence of a faithful right integral, we know that T_1 is injective. Finally observe that

$$\begin{aligned} m(\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1)) &= (\iota \otimes \varphi)(\Delta(a)\Delta(b)(c \otimes 1)) \\ &= \varphi(ab)c. \end{aligned}$$

If we apply all this to a_i, b_i and take the sum, we get the result.

Then, the following definition makes sense.

3.4 Definition Define a linear map $\varepsilon : A \rightarrow \mathbb{C}$ by

$$\varepsilon((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = \varphi(ab)$$

whenever $a, b \in A$.

Recall that every element in A is a sum of elements of the form $(\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$ with $a, b \in A$ (see the proof of Lemma 2.7). So, with this definition, the counit is indeed defined on all of A .

Also notice that, from the proof of the lemma, we get with this definition that indeed $\varepsilon(x)c = \sum p_i q_i$ if $x \otimes c = \sum \Delta(p_i)(1 \otimes q_i)$. This is in accordance with the way ε is originally introduced (see Proposition 3.1). It also shows that the above definition does not depend on the choice of φ .

It also follows immediately from the definition and the faithfulness of φ that $(\varepsilon \otimes \iota)\Delta(a) = a$ for all $a \in A$. Recall that, as we saw in a remark after Definition 2.1, elements of the form $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ are well defined in $M(A)$ for all $a \in A$ and any linear functional ω on A . In particular, this is the case with $\omega = \varepsilon$.

Let us now try to prove the other main properties of the counit.

3.5 Proposition The map $\varepsilon : A \rightarrow \mathbb{C}$ is a homomorphism and satisfies

$$(\varepsilon \otimes \iota)\Delta(a) = a \quad \text{and} \quad (\iota \otimes \varepsilon)\Delta(a) = a$$

for all a . Any other linear map satisfying one of these two equations for all a must be ε .

Proof: There are different ways to obtain that ε is a homomorphism. A cute way is as follows. It is inspired by techniques used in [V-VD].

Take a, b, c in A and use the surjectivity of T_1 to write

$$\begin{aligned} ab \otimes c &= (a \otimes c)(b \otimes 1) = (\sum \Delta(p_i)(1 \otimes q_i))(b \otimes 1) \\ &= \sum \Delta(p_i)(b \otimes q_i) \\ &= \sum \Delta(p_i)\Delta(r_{ij})(1 \otimes s_{ij}) \\ &= \sum \Delta(p_i r_{ij})(1 \otimes s_{ij}). \end{aligned}$$

Then, by the remark following Definition 3.4, we get

$$\begin{aligned} \varepsilon(ab)c &= \sum p_i r_{ij} s_{ij} = \sum p_i \varepsilon(b) q_i \\ &= \varepsilon(b) \sum p_i q_i = \varepsilon(b) \varepsilon(a)c. \end{aligned}$$

This shows that ε is a homomorphism.

We have seen already that $(\varepsilon \otimes \iota)\Delta(a) = a$. Because of the symmetry of our data, we also have a linear map $\varepsilon' : A \rightarrow \mathbb{C}$ satisfying $(\iota \otimes \varepsilon')\Delta(a) = a$ for all a . Now, suppose that ω is any linear map from A to \mathbb{C} satisfying $(\iota \otimes \omega)\Delta(a) = a$. Then we have for all $a, b, c \in A$ that

$$\begin{aligned} (\varepsilon \otimes \iota)((c \otimes 1)\Delta(a)(1 \otimes b)) &= \varepsilon(c)(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) \\ &= \varepsilon(c)ab. \end{aligned}$$

We can cancel b and obtain

$$(\varepsilon \otimes \iota)((c \otimes 1)\Delta(a)) = \varepsilon(c)a$$

for all $a, c \in A$. If we apply ω we get

$$\varepsilon(ca) = \varepsilon(c)\omega(a)$$

for all $a, c \in A$. As $\varepsilon(ca) = \varepsilon(c)\varepsilon(a)$ and ε is non-zero, we get $\varepsilon(a) = \omega(a)$. This not only proves the uniqueness statement but also that $(\iota \otimes \varepsilon)\Delta(a) = a$ because $\varepsilon = \varepsilon'$.

Now we look at the antipode and we proceed in a similar fashion.

Again we first need a lemma.

3.6 Lemma If $\sum(\iota \otimes \varphi)(\Delta(a_i)(1 \otimes b_i)) = 0$, then also $\sum(\iota \otimes \varphi)((1 \otimes a_i)\Delta(b_i)) = 0$.

Proof: As in the proof of Lemma 3.3 we write

$$x \otimes c = T_1((\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1)))$$

where $x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$. Now we apply $\varepsilon \otimes \iota$ to $(\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)(1 \otimes c \otimes 1))$ and we get $(\iota \otimes \varphi)((1 \otimes a)\Delta(b)(c \otimes 1))$.

So, if we apply all this with a_i, b_i , take sums and use the injectivity of T_1 , we get the result.

Now, we can define the antipode.

3.7 Definition Define a linear map $S : A \rightarrow A$ by

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)).$$

Again, as we see in the proof of the lemma, we have that $S(x)c = \sum \varepsilon(p_i)q_i$, where $x \otimes c = \sum \Delta(p_i)(1 \otimes q_i)$. That is the formula for S used in Proposition 3.1. It shows that the antipode does not depend on the choice of φ .

Now we prove the main properties of the antipode. Again there are different ways to do this.

3.8 Proposition We have $S(ab) = S(b)S(a)$ for all $a, b \in A$.

Proof: Take $a, b, c \in A$ and use the surjectivity of the map T_1 . As in the proof of Proposition 3.5 we get

$$\begin{aligned} ab \otimes c &= (a \otimes c)(b \otimes 1) = \sum \Delta(p_i)(b \otimes q_i) \\ &= \sum \Delta(p_i r_{ij})(1 \otimes s_{ij}). \end{aligned}$$

Then

$$\begin{aligned} S(ab)c &= \sum \varepsilon(p_i) \varepsilon(r_{ij}) s_{ij} \\ &= \sum \varepsilon(p_i) S(b) q_i = S(b) S(a) c. \end{aligned}$$

We cancel c and get the result.

3.9 Proposition $(S \otimes S)\Delta(a) = \sigma\Delta(S(a))$ for all $a \in A$.

Proof: We have to give a correct meaning to the above formula. This can be done by multiplying with elements in A or by extending the maps $S \otimes S$ and σ to $M(A \otimes A)$. We will not be very strict however in the following argument because it is quite obvious how to make it precise.

Take $a, b \in A$ and define $x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$. Then $S(x) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b))$ and

$$\begin{aligned} \Delta(S(x)) &= (\iota \otimes \iota \otimes \varphi)((1 \otimes 1 \otimes a)\Delta^{(2)}(b)) \\ &= (\iota \otimes S)(\iota \otimes \iota \otimes \varphi)(\Delta_{23}(a)\Delta_{13}(b)) \\ &= \sigma(S \otimes \iota)(\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)) \\ &= \sigma(S \otimes S)(\iota \otimes \iota \otimes \varphi)(\Delta^{(2)}(a)(1 \otimes 1 \otimes b)) \\ &= \sigma(S \otimes S)\Delta(x). \end{aligned}$$

Finally we have the following.

3.10 Proposition For all $a \in A$ we have

$$\begin{aligned} m(\iota \otimes S)\Delta(a) &= \varepsilon(a)1 \\ m(S \otimes \iota)\Delta(a) &= \varepsilon(a)1. \end{aligned}$$

Moreover, if R is any linear map from A to A , satisfying one of the above formulas, then $R = S$.

Proof: First remark that one has to give a precise meaning to the above formulas. For the first one this is done by multiplying with an element in A from the left. For the second one, it is necessary to multiply with an element from the right. Also in the proof below, one makes things precise in this way.

Take $a, b \in A$. Then

$$\begin{aligned} (\iota \otimes S)\Delta((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) &= (\iota \otimes S)(\iota \otimes \iota \otimes \varphi)(\Delta^{(2)}(a)(1 \otimes 1 \otimes b)) \\ &= (\iota \otimes \iota \otimes \varphi)(\Delta_{13}(a)\Delta_{23}(b)). \end{aligned}$$

So, if we apply multiplication we get

$$m(\iota \otimes S)\Delta((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = (\iota \otimes \varphi)\Delta(a)\Delta(b) = \varphi(ab)1$$

and because $\varepsilon(\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) = \varphi(ab)$, we get the first formula.

By the symmetry of our system, we also have a linear map $S' : A \rightarrow A$ satisfying

$$m(S' \otimes \iota)\Delta(a) = \varepsilon(a)1$$

for all $a \in A$. Now, let R be any linear map from A to A satisfying $m(R \otimes \iota)\Delta(a) = \varepsilon(a)1$ for all a . Then, using the Sweedler notation,

$$\sum R(a_{(1)})a_{(2)}S(a_{(3)}) = \sum \varepsilon(a_{(1)})S(a_{(2)}) = S(a)$$

and also

$$\sum R(a_{(1)})a_{(2)}S(a_{(3)}) = \sum R(a_{(1)})\varepsilon(a_{(2)}) = R(a).$$

We get $R = S$ and in particular $S = S'$. So we have also proven the second formula as well the uniqueness property.

The last part of this proof is standard but again it has to be done correctly by multiplying with elements in A (because we are working with multiplier Hopf algebras).

Let us finish this section first with a remark on the $*$ -algebra case. When we have a $*$ -algebra with a comultiplication Δ , we assume that Δ is a $*$ -homomorphism. Usually, we also require the integrals to be positive (see section 2). For the moment, let us not make this requirement. On the other hand, it is always possible to take the integrals self-adjoint

(in the sense that $\varphi(a^*) = \overline{\varphi(a)}$ for all a). Then, it is easily seen from the definitions that ε is a *-homomorphism whereas for the antipode, we get $S(a^*) = (S^{-1}(a))^*$. In fact, these results also follow easily from the uniqueness and the other properties of the counit and the antipode we have proven here.

We do not prove the other known properties of multiplier Hopf algebras with integrals (like the uniqueness of the integrals, the weak K.M.S. property, ...). We refer to the original paper [VD5] for these results.

4. The case of a cointegral

In their original paper, Larson and Sweedler considered a pair of an algebra A and a comultiplication Δ with the existence of a left cointegral. For a comultiplication with a counit ε , a left cointegral is defined as a non-zero element $h \in A$ satisfying $ah = \varepsilon(a)h$ for all $a \in A$.

We would like to start without the assumption of the existence of a counit, as we did also in Section 2 where we proved the main theorem. Therefore, we will start with another definition.

Again, A is an algebra over \mathbb{C} with a non-degenerate product and $\Delta : A \rightarrow M(A \otimes A)$ is a comultiplication on A (Definition 1.1). We will assume right away that Δ is regular (Definition 1.2).

4.1 Definition A non-zero element $h \in A$ is called a *left cointegral* if $\Delta(a)(1 \otimes h) = a \otimes h$ for all $a \in A$.

If there is a counit ε , then this condition is equivalent with $ah = \varepsilon(a)h$ for all $a \in A$. In fact, it is not so hard to show the existence of a counit in the following sense.

4.2 Proposition Assume that h is a left cointegral as in Definition 4.1. If the right leg of Δ is all of A , then there is a homomorphism $\varepsilon : A \rightarrow \mathbb{C}$ such that $ah = \varepsilon(a)h$ and $(\iota \otimes \varepsilon)\Delta(a) = a$ for all $a \in A$.

Proof: From $\Delta(a)(1 \otimes h) = a \otimes h$ it follows that $bh \in \mathbb{C}h$ for all elements b in the right leg of $\Delta(a)$. So, by our assumption, $Ah = \mathbb{C}h$. Hence, there is a linear map $\varepsilon : A \rightarrow \mathbb{C}$ defined by $ah = \varepsilon(a)h$.

We have $\varepsilon(ab)h = abh = \varepsilon(b)ah = \varepsilon(a)\varepsilon(b)h$ and so $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in A$. Moreover

$$a \otimes h = \Delta(a)(1 \otimes h) = (\iota \otimes \varepsilon)\Delta(a) \otimes h$$

so that $(\iota \otimes \varepsilon)\Delta(a) = a$ for all a .

We also have the following.

4.3 Proposition Again assume that the right leg of Δ is all of A and let $\varepsilon : A \rightarrow \mathbb{C}$ be the homomorphism constructed in the previous proposition. We also have $(\varepsilon \otimes \iota)\Delta(a) = a$ for all $a \in A$ and $\Delta(a)(h \otimes 1) = h \otimes a$.

Proof: For all $a \in A$ we have

$$(\iota \otimes \varepsilon \otimes \iota)\Delta^{(2)}(a) = \Delta(a).$$

As the right leg of Δ is assumed to be all of A , this formula implies that $(\varepsilon \otimes \iota)\Delta(b) = b$ for all $b \in A$.

Then $\Delta(a)(h \otimes 1) = h \otimes (\varepsilon \otimes 1)\Delta(a) = h \otimes a$ for all $a \in A$.

Observe that the formula $\Delta(a)(1 \otimes h) = a \otimes h$ implies that a belongs to the left leg of $\Delta(a)$. So the left leg of Δ is all of A . If we now also assume that the right leg of Δ is all of A , then Δ is full (Definition 1.8). So, for a full comultiplication the equality $\Delta(a)(1 \otimes h) = a \otimes h$ for all a implies that also $\Delta(a)(h \otimes 1) = h \otimes a$. We also have a counit ε and h is a cointegral in the original sense.

Let us mention here that the situation above is very similar to the one encountered in some notes that were written in 1993 (see [VD1]). We have been inspired by some of the techniques used in these notes.

In what follows, we will now assume that the comultiplication is full.

As in the case of integrals (Section 2), we can use the cointegrals to show that the four maps T_1, T_2, T'_1, T'_2 are bijective. We need the cointegrals to be 'faithful':

4.4 Definition A cointegral h is called *faithful* if both the left and the right leg of $\Delta(h)$ are all of A .

Observe that this notion is in accordance with the notion of faithfulness for a linear functional (Definition 2.2) when we consider h as a linear functional on the (reduced) dual. In the finite-dimensional case, one condition is sufficient. If e.g. the left leg of $\Delta(h)$ is all of A , then the same is true for the right leg. Compare with the remark following Definition 2.2.

Also remark that, when there exists a faithful left cointegral, it is automatic that the comultiplication is full.

Now, we have the following analogues of Lemma 2.4 and Lemma 2.7.

4.5 Lemma If h is a left cointegral such that the left leg of $\Delta(h)$ is all of A , then the map T_2 is injective.

Proof: Take $a \in A$, apply $\Delta \otimes \iota$ to the equation $h \otimes a = \Delta(a)(h \otimes 1)$ and multiply with $b \in A$ to get

$$\begin{aligned} (b \otimes 1 \otimes a)(\Delta(h) \otimes 1) &= (b \otimes 1 \otimes 1)\Delta^{(2)}(a)(\Delta(h) \otimes 1) \\ &= (\iota \otimes \Delta)((b \otimes 1)\Delta(a))(\Delta(h) \otimes 1). \end{aligned}$$

So, if $\sum(b_i \otimes \iota)\Delta(a_i) = 0$, we get

$$\sum(b_i \otimes 1 \otimes a_i)(\Delta(h) \otimes 1) = 0.$$

By assumption, the left leg of $\Delta(h)$ is all of A , so $\sum b_i c \otimes a_i = 0$ for all $c \in A$. Therefore $\sum b_i \otimes a_i = 0$. This proves the result.

Doing the same thing for the other leg, and on the other side (flip Δ , flip the product, or flip both), we get:

4.6 Proposition If there exist a faithful left integral and a faithful right integral, then the four maps

$$\begin{array}{ll} a \otimes b \mapsto \Delta(a)(1 \otimes b) & a \otimes b \mapsto \Delta(a)(b \otimes 1) \\ a \otimes b \mapsto (a \otimes 1)\Delta(b) & a \otimes b \mapsto (1 \otimes a)\Delta(b) \end{array}$$

are injective.

What about surjectivity?

4.7 Lemma If the left leg of $\Delta(h)$ is all of A , then T_1 is surjective.

Proof: Now apply $\iota \otimes \Delta$ to $a \otimes h = \Delta(a)(1 \otimes h)$ and multiply with $1 \otimes 1 \otimes b$ to get

$$a \otimes (\Delta(h)(1 \otimes b)) = \Delta^{(2)}(a)(1 \otimes (\Delta(h)(1 \otimes b))),$$

write $\Delta(h)(1 \otimes b) = \sum u_i \otimes v_i$ and apply a linear functional ω (on the third leg of this equation) to get

$$a \otimes x = \sum \Delta(p_i)(1 \otimes u_i)$$

where $x = (\iota \otimes \omega)(\Delta(h)(1 \otimes b))$ and $p_i = (\iota \otimes \omega)(\Delta(a)(1 \otimes v_i))$. This proves the lemma.

Similarly, we find:

4.8 Proposition If there exist a faithful left integral and a faithful right integral then the four maps

$$\begin{array}{ll} a \otimes b \mapsto \Delta(a)(1 \otimes b) & a \otimes b \mapsto \Delta(a)(b \otimes 1) \\ a \otimes b \mapsto (a \otimes 1)\Delta(b) & a \otimes b \mapsto (1 \otimes a)\Delta(b) \end{array}$$

are all surjective.

So, we get the following generalization of the Larson-Sweedler result.

4.9 Theorem Let A be an algebra over \mathbb{C} with a non-degenerate product. Let Δ be a regular comultiplication on A . Assume that there are faithful left and right cointegrals. Then A is a regular multiplier Hopf algebra with integrals (of discrete type).

Proof: By Proposition 4.6 and 4.8, all the four maps T_1, T_2, T'_1, T'_2 are bijective. Hence we have a regular multiplier Hopf algebra. Because there exist cointegrals, there also exist integrals (see [VD3] and [VD-Z1]).

When we look at the special case of a finite-dimensional algebra, we recover (a slightly stronger form of) the original version of the Larson-Sweedler theorem:

4.10 Theorem Let A be a finite-dimensional algebra with 1 . Let Δ be a comultiplication on A . Assume that there is a faithful left cointegral. Then A is a Hopf algebra.

Proof: By Lemma 4.5 the map T_2 is injective and so also bijective as A is finite-dimensional. By Lemma 4.7 the map T_1 is surjective and so also bijective. Therefore A is a Hopf algebra.

Recall that the existence of a faithful left cointegral (in the sense of Definition 4.1) gives us the existence of a counit. The cointegral is also a cointegral in the usual sense. In the original form of the theorem of Larson and Sweedler, the existence of a counit is part of the assumptions. It is also interesting to compare Theorem 4.10 with Theorem 2.9.

To finish this section, let us indicate how it is possible to develop the theory, starting from the assumptions in Theorem 4.9. Because this case is less general than the case studied in Section 2, we will not do this in detail as we have done in Section 3. We will just briefly give some indications.

One method would be to start with the assumptions in Theorem 4.9, use the result and pass immediately to the dual. Then one can proceed as in Section 3. A more direct approach would be as follows.

Let A be an algebra over \mathbb{C} with a non-degenerate product and let Δ be a regular coproduct on A . Assume as in Theorem 4.9 that there is a faithful left cointegral and a faithful right cointegral. We have seen in Proposition 4.2 and Proposition 4.3 how to find the counit and its properties. What about the antipode? The following property (which is well-known) is dual to the formula in Proposition 3.2 in the case of integrals and again, it is crucial for this point of view. Remark that it was also used in [VD1] to construct the antipode.

4.11 Proposition Let (A, Δ) be a multiplier Hopf algebra with a left cointegral h . Then

$$(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$$

for all $a \in A$.

Proof: Using the Sweedler notation, we get for all $a \in A$ that

$$\begin{aligned} (1 \otimes a)\Delta(h) &= (S(a_{(1)})a_{(2)} \otimes a_{(3)})\Delta(h) \\ &= (S(a_{(1)}) \otimes 1)\Delta(a_{(2)}h) \\ &= (\varepsilon(a_{(2)})S(a_{(1)}) \otimes 1)\Delta(h) \\ &= (S(a) \otimes 1)\Delta(h) \end{aligned}$$

and this proves the result.

Because of the faithfulness of h , the left leg of $\Delta(h)$ is all of A . Now, it is not so difficult to show that the result in Proposition 4.11 can be used to define the antipode and prove its properties. We leave it to the reader as an excercise.

And just a small remark to finish. The result in Proposition 4.11 can be used to give a simple argument to show that the antipode is automatically bijective in the case of a finite-dimensional Hopf algebra. Indeed, if $S(a) = 0$ it follows from the above formula and the fact that the right leg of $\Delta(h)$ is all of A , that $a = 0$.

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